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ON THE BRST QUANTIZATION OF CHIRAL BOSONS

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ABSTRACT

The Siegel action for two right-moving chiral bosons can be BRST quantized. In the case in which their kinetic terms have opposite signs the momenta in the left-moving sector must be constrained to be zero. In this case the two chiral bosons can be described also by a quadratic action. There is no analogous BRST quantization for a single chiral boson.

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1. Introduction

Self-dual fields play an important role in superstring theory. They were studied first in connection with the type IIB superstring, and the corresponding $D = 10$, $N = 2$ chiral supergravity[1]; they appear also in the minimal $D = 6$ supergravity[2]. Finally, they were an essential ingredient in the construction of the heterotic string[3].

A covariant lagrangian quantization of self-dual fields is not known. Some quantum properties of self-dual fields have been studied considering systems involving fermions. Examples of this are the computation of the gravitational anomaly associated to self-dual fields using bispinors[4], or, in $D = 2$, the use of fermionization[5]. It is therefore an interesting problem to find a covariant lagrangian formulation for self-dual fields to better study their quantum properties. It has been shown[2] that there is no Lorentz covariant quadratic lagrangian formulation for a self-dual field in $D = 6$ and 10. This is related to the fact that the self-duality condition is a second-class constraint and so if one introduces a Lagrange multiplier in the lagrangian to obtain this constraint as an equation of motion, one can not gauge away the Lagrange multiplier from the equations of motion. To overcome this difficulty one can try to substitute the second-class constraint by a first-class constraint. To our knowledge, there are two known ways to achieve this: either one can square the second-class constraint to obtain a first-class constraint, approach followed by Siegel[6]; or one can introduce an extra field together with a first class constraint[7]. We will see that there is a relationship between these two approaches.

Siegel has shown[6] that squaring the self-duality constraint in $D = 2$ and $D = 6$ one gets classically a first-class constraint, leading to a truncated conformal gravity. The truncated gravity field is a Lagrange multiplier and a gauge field; its equation of motion gives (in an axial-like gauge) a first-class constraint which can be solved in the classical theory in terms of the second-class constraint corresponding to the self-duality of the boson. An analogous formulation[8] has been studied in $D = 10$.

We study the BRST quantization of the Siegel lagrangian[6] in $D = 2$. The self-dual fields in this case are scalars $\phi_i(\tau, \sigma)$, $i = 1, ..., n$. We find that the gauge field $\lambda^{++}(\tau, \sigma)$ has an anomaly, analogous to the gravitational anomaly[4]. To cancel this anomaly we introduce a 'Liouville term'. We BRST quantize this theory following the procedure in ref.[9,10]. We construct the nilpotent BRST operator Q and we impose the condition $Q|\Phi >= 0$ on the physical states. If $n = 1$, the 'intercept' of the operator Q is negative, and there are no physical states. If $n = 2$, the 'intercept' is zero, and only the right-moving modes of the two scalars are physical states. In this case we prove the no-ghost theorem in the left-moving sector both in the case in which the scalars have the same sign in the kinetic term ('Euclidean case'), and in the case in which they have the opposite sign ('Minkowski case'), the former case being the one physically interesting[3]. The Q-physical states in the 'Euclidean case' are those described by two right-moving bosons; the Q-physical states in the 'Minkowski case' are described by two right-moving bosons and by the null momenta of the left-moving sector; an extra constraint must be imposed to eliminate these momenta to describe two chiral bosons in the 'Minkowski case'. The main reason for studying the latter case is that the no-ghost theorem is not trivially satisfied, requiring an analysis of the KO[9] quartet similar to the one done for the string[10]. This analysis shows that the first-class constraint can consistently eliminate two left-moving bosons. Wick rotating the left-moving momenta one relates the no-ghost theorems in the two cases. Therefore we conclude that the Siegel action for two chiral bosons can be BRST quantized. We show that for $n > 2$ the Siegel action does not describe chiral bosons. To describe $2n$ chiral bosons one can use n gauge fields.

We then proceed to consider the generalized hamiltonian formalism for second-class constraints of Batalin and Fradkin[7] and we apply it to the chiral boson. This formulation transforms any second-class constraint on the original canonical variables into a first-class constraint while introducing an extra propagating field. Applying this formalism to the chiral boson one obtains, after going to the lagrangian formalism, a quadratic lagrangian which contains fields $\phi(\tau, \sigma)$ and $\zeta(\tau, \sigma)$ with opposite sign in their kinetic terms and possesses a gauge invariance under $\delta\phi = \delta\zeta = \epsilon$. The resulting system has also a self-duality constraint on the extra field ζ. Dropping the self-duality constraint on ζ, this lagrangian describes two chiral bosons in the 'Minkowski case', provided one requires the left-moving momentum of ζ to vanish. The no-ghost theorem is proved in the BRST quantization, leading to the same cohomologically non-trivial states as in the Siegel action, *i.e.*, states built out of right-moving modes of $\phi(\tau, \sigma)$ and $\zeta(\tau, \sigma)$. This lagrangian can be easily coupled to gravity, leading to the correct contribution to the conformal anomaly due to two chiral bosons. This quadratic lagrangian is analogous to a gauged Wess-Zumino model with an abelian Kac-Moody symmetry[11]; the former cubic lagrangian has the Virasoro symmetry corresponding to this Kac-Moody symmetry.

We comment that it is not possible to eliminate altogether a single self-duality second-class constraint in favor of a first-class constraint; at most the second-class constraint is transferred from one sector of the phase space to another. The reason for this is that the second-class constraint gives the square-root of a determinant in the path-integral, while a first-class constraint gives a determinant. Only paired second-class constraints can be eliminated in favor of first-class constraints. This is the reason why one can BRST quantize two chiral bosons.

The paper is organized in the following way. In sect. 2 we quantize the Siegel lagrangian and we prove the no-ghost theorem. In sect. 3 we introduce a quadratic lagrangian for chiral bosons, then we construct the Batalin-Fradkin hamiltonian for a chiral bosons and we establish its relation with the quadratic lagrangian. Finally, in sect. 4 we state our conclusions.

2. Quantization of the Siegel Lagrangian

A right-moving scalar field $\phi(\tau,\sigma)$ in $D=2$ is described by the lagrangian†

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi\eta^{\alpha\beta} \\ &= -\partial_+\phi\partial_-\phi,\end{aligned} \tag{2.1}$$

with the constraint

$$\partial_+\phi \approx 0. \tag{2.2}$$

Canonically quantizing this system one introduces the canonical conjugate momentum $\pi(\tau,\sigma)$ and the equal time commutation relation

$$[\phi(\sigma),\pi(\sigma')] = i\delta(\sigma-\sigma'), \tag{2.3}$$

where the τ-dependence of the fields has been omitted. Constraint (2.2) is now given by

$$T(\sigma) = \frac{1}{\sqrt{2}}(\pi(\sigma)+\phi'(\sigma)), \tag{2.4}$$

† In our conventions: $\eta^{\sigma\sigma}=-\eta^{\tau\tau}=1$, $\eta^{\tau\sigma}=\eta^{\sigma\tau}=0$, $\partial_\pm=\frac{1}{\sqrt{2}}(\partial_\sigma\pm\partial_\tau)$, $\eta^{+-}=\eta^{-+}=1$, $\eta^{++}=\eta^{--}=0$.

where we have used the notation $\phi'(\sigma) = \partial_\sigma \phi(\sigma)$. The algebra of the constraints is

$$[T(\sigma), T(\sigma')] = i\partial_\sigma \delta(\sigma - \sigma'), \tag{2.5}$$

and so they are second class. The second-class constraints can be substituted by the first class constraints

$$L(\sigma) = \frac{1}{4} : T(\sigma)^2 :, \tag{2.6}$$

with the algebra

$$[L(\sigma), L(\sigma')] = \frac{i}{2}\partial_\sigma \delta(\sigma - \sigma')(L(\sigma) + L(\sigma')) - \frac{i}{96\pi}\partial_\sigma^3 \delta(\sigma - \sigma'). \tag{2.7}$$

$L(\sigma)$ are Virasoro generators satisfying an anomalous algebra. From string theory it is well known that the Virasoro constraints can be realized in lagrangian form with an auxiliary gravity field. Since in our case the generators are only in the left-moving sector, only half of the modes of gravity will be needed. Following Siegel[6], let us consider a bosonic field $\phi(\tau, \sigma)$ propagating in a background gravity field $g_{\alpha\beta}$,

$$\mathcal{L}_1 = -\frac{1}{2}\sqrt{g} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \tag{2.8}$$

$$\delta g^{\alpha\beta} = -(\partial_\gamma \epsilon^\alpha) g^{\beta\gamma} - (\partial_\gamma \epsilon^\beta) g^{\alpha\gamma} + (\partial_\gamma g^{\alpha\beta})\epsilon^\gamma, \tag{2.9}$$

$$\delta\phi = \epsilon^\alpha \partial_\alpha \phi. \tag{2.10}$$

Truncating the auxiliary gravity field $g_{\alpha\beta}$,

$$\sqrt{g} g^{\alpha\beta} = \begin{pmatrix} -\lambda^{++} & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.11}$$

one finds,

$$\mathcal{L}_1 = -\partial_+ \phi \partial_- \phi + \frac{1}{2}\lambda^{++}(\partial_+ \phi)^2, \tag{2.12}$$

i.e. the field λ^{++} is the doubly self-dual gauge field realizing constraint (2.6). The lagrangian is invariant under the transformations

$$\delta\phi = \epsilon^+ \partial_+ \phi, \tag{2.13a}$$

$$\delta\lambda^{++} = 2\partial_- \epsilon^+ + \epsilon^+ \partial_+ \lambda^{++} - \lambda^{++} \partial_+ \epsilon^+. \tag{2.13b}$$

The transformation laws (2.13) follow from reparametrization invariance, considering only one of the reparametrization parameters.

An alternative way to obtain (2.12) is to notice that (2.1) has the conformal invariance

$$\delta\phi = \epsilon^+ \partial_+ \phi, \qquad \partial_- \epsilon^+ = 0. \tag{2.14}$$

The part of the variation of the lagrangian contributing to the Noether current of this rigid symmetry is

$$\delta\mathcal{L} = -(\partial_- \epsilon^+)(\partial_+ \phi)^2. \tag{2.15}$$

Therefore, the field λ^{++} can be considered as the gauge field for the gauging of the conformal invariance (2.14). This procedure can be realized also in the presence of a gravity field $h_{\alpha\beta}$. Consider

$$\mathcal{L}_h = -\frac{1}{2}\sqrt{h} h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \tag{2.16}$$

and the transformations

$$\begin{aligned} \delta\phi &= \epsilon^\alpha \partial_\alpha \phi, \\ \delta h_{\alpha\beta} &= 0. \end{aligned} \tag{2.17}$$

Under these transformations, the variation of the lagrangian (2.13) becomes

$$\delta\mathcal{L}_h = -\sqrt{h}(D^\alpha \epsilon^\beta - \frac{1}{2} h^{\alpha\beta} D_\gamma \epsilon^\gamma) \partial_\alpha \phi \partial_\beta \phi, \tag{2.18}$$

where D^α are covariant derivatives respect to the metric $h^{\alpha\beta}$. If ϵ^α is a conformal killing vector, *i.e.*, if it satisfies

$$D_\alpha \epsilon_\beta + D_\beta \epsilon_\alpha - h_{\alpha\beta} D^\gamma \epsilon_\gamma = 0, \tag{2.19}$$

then (2.17) is a symmetry of the lagrangian (2.16). In particular, for a flat metric, (2.19) implies that the killing vectors are of the form† $\epsilon^+ = \epsilon^+(\sigma^+)$ and $\epsilon^- = \epsilon^-(\sigma^-)$. Gauging one of the symmetries in (2.17), one finds[3], after introducing vielbiens e^α_μ,

$$\mathcal{L}_h = -\frac{1}{2}\sqrt{h}\Big(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \lambda^{++} e^\alpha_+ e^\beta_+ \partial_\alpha\phi\partial_\beta\phi \Big). \tag{2.20}$$

Besides being reparametrization invariant, this lagrangian is invariant under the transformation laws

$$\begin{aligned} \delta\phi &= \epsilon^+ \mathcal{D}_+ \phi, \\ \delta\lambda^{++} &= 2\mathcal{D}_- \epsilon^+ + \epsilon^+ \mathcal{D}_+ \lambda^{++} - \lambda^{++} \mathcal{D}_+ \epsilon^+, \\ \delta h_{\alpha\beta} &= 0, \end{aligned} \tag{2.21}$$

where $\mathcal{D}_\pm = e^\alpha_\pm D_\alpha$ and $\epsilon_- = e^\alpha_- \epsilon_\alpha$. Similarly, one could have proceeded for the case of gauging the symmetry associated to ϵ_+.

To have a consistent quantum theory the anomalies must cancel. Going back to (2.12) we observe that the gauge symmetry (2.13) is anomalous; the anomaly is essentially a gravitational-like anomaly[4]. The partition function corresponding to the lagrangian (2.12) involves an integration over the field ϕ as well as over the gauge field λ^{++}. However, let us consider for the moment the effective action $\Gamma(\lambda^{++})$ arising when one only integrates over the ϕ field. The anomalous term of this effective action is

$$\Gamma(\lambda^{++}) = k^2 \int d^2\sigma \lambda^{++}(\sigma) \frac{\partial^3_\pm}{\partial_-} \lambda^{++}(\sigma) + O(\lambda^3), \tag{2.22}$$

† In our conventions: $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma \pm \tau)$.

where $k^2 = \frac{1}{192\pi}$. To make the theory consistent quantum mechanically we must cancel the anomaly. This situation is similar to having a bosonic string in less than 26 dimensions. In that case Polyakov[12] introduced a Liouville field as an extra propagating field to cancel the anomaly. Let us try to use a similar mechanism in our case. Consider a 'truncated gravity field' of the form

$$g^{\alpha\beta} = \begin{pmatrix} g^{++} & g^{+-} \\ g^{-+} & 0 \end{pmatrix}, \tag{2.23}$$

or, after defining,

$$\rho = g^{+-} = g^{-+}, \tag{2.24a}$$

$$\lambda^{++} = -\frac{g^{++}}{\rho}, \tag{2.24b}$$

of the form,

$$\sqrt{g} g^{\alpha\beta} = \begin{pmatrix} -\lambda^{++} & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.25}$$

Equation (2.25) is the same as (2.11). However, this truncated gravity field has an extra (Weyl) mode $\sqrt{g} = \rho^{-1}$, transforming under the gauge symmetry (reparametrizations) as

$$\delta\rho = \epsilon^+ \partial_+ \rho - \rho \partial_+ \epsilon^+. \tag{2.26}$$

After defining

$$\mu = 2k \log \rho, \tag{2.27}$$

where k is the c-number in (2.22), one finds, from (2.26),

$$\delta\mu = \epsilon^+ \partial_+ \mu - 2k \partial_+ \epsilon^+. \tag{2.28}$$

Notice that λ^{++} as defined in (2.24b) transforms as in (2.13b).

The scalar curvature corresponding to (2.25) has the form

$$
\begin{aligned}
2k\sqrt{g}R &= \partial_+\Big(-2\partial_-\mu + \lambda^{++}\partial_+\mu - 2k\partial_+\lambda^{++}\Big) \\
&= -\sqrt{g}\Box\mu - 2k\partial_+\partial_+\lambda^{++},
\end{aligned} \tag{2.29}
$$

where $\Box$ is the covariantized d'Alambertian. Consider the following term

$$
2k^2\int d\tau d\sigma d\tau' d\sigma' (\sqrt{g}R)(\sigma)G(\tau,\sigma;\tau',\sigma')(\sqrt{g}R)(\sigma'), \tag{2.30}
$$

where $G(\tau,\sigma;\tau',\sigma')$ is the covariant Green function satisfying,

$$
\sqrt{g}\Box G(\tau,\sigma;\tau',\sigma') = \delta(\tau-\tau')\delta(\sigma-\sigma'). \tag{2.31}
$$

The term (2.30) is equal to the anomalous contribution of the effective action (2.22) plus the local term:

$$
\Delta\mathcal{L}_1 = -\Big(\partial_+\mu\partial_-\mu - \frac{1}{2}\lambda^{++}(\partial_+\mu)^2 + 2k(\partial_+\lambda^{++})(\partial_+\mu)\Big) \tag{2.32}
$$

The variation of (2.32) under (2.28) and (2.13b) is,

$$
\delta\Delta\mathcal{L}_1 = -4k^2(\partial_+\partial_+\partial_+\lambda^{++})\epsilon^+, \tag{2.33}
$$

which cancels the anomalous contribution coming from the variation of (2.22) at the quantum level. Therefore, the effective action $\Gamma(\lambda^{++},\mu)$ arising from the lagrangian made out of (2.12) and (2.32) is anomaly free.

This analysis of the effective action $\Gamma(\lambda^{++},\mu)$ motivates one of the modifications of the Siegel lagrangian (2.12) that one must make to have a consistent quantum theory. One observes that a term like the last one in (2.32) could be added to (2.12), *i.e.* a term of the form $k_\phi(\partial_+\lambda^{++})(\partial_+\phi)$. On the other hand, (as we will show in detail below) in order to have a BRST quantized theory one must add a new scalar $\zeta(\tau,\sigma)$. In computing the partition function of the theory one must integrate over ϕ, λ^{++} and ζ. This means that one must take into account the contributions to the gauge anomaly of each of these fields. Let us keep the sign η of the kinetic term of ζ not fixed. From ϕ and ζ one has two types of contributions: the one-loop-contribution, which is as the one leading to (2.22) and is the same for both fields, and the tree-level-contribution which comes from terms in the lagrangian of the form $k_\phi(\partial_+\lambda^{++})(\partial_+\phi)$ or $k_\zeta(\partial_+\lambda^{++})(\partial_+\zeta)$. The contribution to the gauge anomaly from these last terms depends on the sign of the kinetic term of the corresponding field. This fact singles out which terms should be introduced in the modified lagrangian. Finally, from the ghosts of the gauge field one obtains a contribution that is -26 times the one of ϕ. In summary, one has the following contributions to the gauge anomaly:

ϕ − one-loop	$\frac{1}{192\pi}$
ζ − one-loop	$\frac{1}{192\pi}$
ϕ − tree-level	$\frac{k_\phi^2}{4}$
ζ − tree-level	$-\eta\frac{k_\zeta^2}{4}$
λ^{++}	$-\frac{26}{192\pi}$

The minimal lagrangian which gives a theory free of anomalies and includes the extra right-moving mode ζ is the one made out of only one 'k-term', namely, the one associated to ϕ. We propose the modified Siegel lagrangian:

$$\begin{aligned}\tilde{\mathcal{L}}_1 =& -\partial_+\phi\partial_-\phi + \eta\partial_+\zeta\partial_-\zeta + \frac{1}{2}\lambda^{++}\Big((\partial_+\phi)^2 - \eta(\partial_+\zeta)^2\Big)\\ &-\frac{1}{\sqrt{2\pi}}(\partial_+\lambda^{++})(\partial_+\phi).\end{aligned} \tag{2.34}$$

Notice that $\eta = 1$ corresponds to the 'Minkowski case' and $\eta = -1$ to the 'Euclidean case'. The non-minimal version of this lagrangian has the last term in (2.34) replaced by

$-\frac{1}{\sqrt{2\pi}}(\partial_+\lambda^{++})(\omega_1\partial_+\phi - \eta\omega_2\partial_+\zeta)$ with ω_1 and ω_2 real parameters such that $\omega_1^2 - \eta\omega_2^2 = 1$. This observation can be concluded from the invariance of the first four terms of (2.34) under Lorentz transformations ($\eta = 1$) or rotations ($\eta = -1$) which treat ϕ and ζ as coordinates. Notice also that according to the contributions to the gauge anomaly listed above these non-minimal versions are anomaly free. The lagrangian (2.34) corresponds to the simplest choice $\omega_2 = 0$.

The lagrangian (2.34) is invariant, at the quantum level, under the symmetry transformations

$$\delta\phi = \epsilon^+\partial_+\phi - \frac{1}{\sqrt{2\pi}}\partial_+\epsilon^+, \tag{2.35a}$$

$$\delta\zeta = \epsilon^+\partial_+\zeta, \tag{2.35b}$$

$$\delta\lambda^{++} = 2\partial_-\epsilon^+ + \epsilon^+\partial_+\lambda^{++} - \lambda^{++}\partial_+\epsilon^+. \tag{2.35c}$$

In the non-minimal versions of lagrangian (2.34) the transformation (2.35a) has a factor ω_1 multiplying the last term, and the transformation (2.35b) gets the additional term $-\eta\frac{\omega_2}{\sqrt{2\pi}}\partial_+\epsilon^+$.

To perform the BRST quantization of this theory let us substitute ϵ^+ by Λc^+, where c^+ is a ghost field and Λ a constant anticommuting parameter. The BRST transformations of the fields ϕ and ζ are,

$$\delta_{\rm BRST}\phi = \Lambda\Big(c^+\partial_+\phi - \frac{1}{\sqrt{2\pi}}\partial_+c^+\Big), \tag{2.36a}$$

$$\delta_{\rm BRST}\zeta = \Lambda c^+\partial_+\zeta, \tag{2.36b}$$

$$\delta_{\rm BRST}\lambda^{++} = \Lambda\Big(2\partial_-c^+ + c^+\partial_+\lambda^{++} - \lambda^{++}\partial_+c^+\Big). \tag{2.36c}$$

Following ref.[10] we introduce the gauge fixing $\lambda^{++} = 0$. The quantum lagrangian is given by

$$\tilde{\mathcal{L}}_1^q = \tilde{\mathcal{L}}_1 - \frac{i}{2}\hat{\delta}(b_{++}\lambda^{++}), \tag{2.37}$$

where the symbol $\hat{\delta}$ refers to the BRST transformations (2.36) without the parameter Λ and b_{++} and $\mathcal{B}_{++}$ are the ghost and auxiliary fields, respectively. Their BRST transformations are

$$\begin{aligned} \delta_{\mathrm{BRST}} b_{++} &= \Lambda \mathcal{B}_{++}, \\ \delta_{\mathrm{BRST}} \mathcal{B}_{++} &= 0. \end{aligned} \tag{2.38}$$

The lagrangian (2.37) is invariant under BRST transformations from the quantum point of view, *i.e.*, including the anomalous contribution coming from the BRST transformation of the anomalous terms like (2.22). In a way similar to ref.[10] one can eliminate all the terms in $\tilde{\mathcal{L}}_1^q$ containing the gauge field λ^{++} by shifting $\mathcal{B}_{++}$, apart from the term $\mathcal{B}'_{++}\lambda_{++}$. Since $\mathcal{B}'_{++}$ and λ^{++} appear only in this term in the lagrangian one can eliminate them obtaining the quantum lagrangian,

$$\tilde{\mathcal{L}}_1^q = -\partial_+\phi\partial_-\phi + \eta\partial_+\zeta\partial_-\zeta + ib_{++}\partial_- c^+, \tag{2.39}$$

and the BRST transformations are given by (2.36) and

$$\begin{aligned} &\delta_{\mathrm{BRST}} c^+ = \Lambda c^+ \partial_+ c^+, \\ &\delta_{\mathrm{BRST}} b_{++} = -i\Lambda\Big((\partial_+\phi)^2 - \eta(\partial_+\zeta)^2 + \frac{2}{\sqrt{2\pi}}\partial_+\partial_+\phi - 2ib_{++}\partial_+ c^+ + ic^+\partial_+ b_{++})\Big). \end{aligned} \tag{2.40}$$

Notice that (2.39) is invariant under (2.40) and (2.36a-b). This may look surprising since (2.34) is not invariant under (2.36) and (2.38) at the classical level. However, the terms in (2.34) responsible for the non-invariance have been shifted away with the term $B'_{++}\lambda^{++}$. The conserved BRST current is

$$J^-_{\mathrm{BRST}} = c^+\Big(- (\partial_+\phi)^2 + \eta(\partial_+\zeta)^2 - \frac{2}{\sqrt{2\pi}}\partial_+\partial_+\phi + ib_{++}\partial_+ c^+\Big), \tag{2.41}$$

while $J^+_{\rm BRST}$ vanishes on-shell. From (2.41) the expression of the BRST charge is:

$$Q_1 = \frac{\sqrt{2\pi}}{2}\int d\sigma c^+\Big((\partial_+\phi)^2 - \eta(\partial_+\zeta)^2 + \frac{2}{\sqrt{2\pi}}\partial_+\partial_+\phi - ib_{++}\partial_+ c^+\Big). \qquad (2.42)$$

We perform the canonical quantization on the circle and we use the Fourier expansions:

$$\begin{aligned}
\phi(\tau,\sigma) &= \phi_0 + \frac{1}{2\pi}p_\phi(\tau+\sigma) + \frac{1}{2\pi}\tilde{p}_\phi(\tau-\sigma) + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}(\tilde{\alpha}_n \mathrm{e}^{in(\sigma-\tau)} + \alpha_n \mathrm{e}^{-in(\sigma+\tau)}),\\
\zeta(\tau,\sigma) &= \zeta_0 + \frac{1}{2\pi}p_\zeta(\tau+\sigma) + \frac{1}{2\pi}\tilde{p}_\zeta(\tau-\sigma) + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{1}{n}(\tilde{\beta}_n \mathrm{e}^{in(\sigma-\tau)} + \beta_n \mathrm{e}^{-in(\sigma+\tau)}),\\
b_{++}(\tau,\sigma) &= -\frac{1}{\sqrt{\pi}}(b_0 + \sum_{n\neq 0} b_n \mathrm{e}^{-in(\sigma+\tau)}),\\
c^+(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}(c_0 + \sum_{n\neq 0} c_n \mathrm{e}^{-in(\sigma+\tau)}),
\end{aligned} \qquad (2.43)$$

where

$$\begin{aligned}
&[\phi_0, p_\phi] = [\phi_0, \tilde{p}_\phi] = \frac{i}{2} & &[\zeta_0, p_\zeta] = [\zeta_0, \tilde{p}_\zeta] = -\frac{i}{2}\\
&[\alpha_m, \alpha_n] = m\delta_{m+n} & &[\beta_m, \beta_n] = -m\delta_{m+n}\\
&[\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n} & &[\tilde{\beta}_m, \tilde{\beta}_n] = -m\delta_{m+n}\\
&\{b_m, c_n\} = \delta_{m+n} & &
\end{aligned} \qquad (2.44)$$

Using these expansions and (2.42) we obtain

$$Q_1 = \frac{1}{2}\sum_n \Big(: c_{-n}\sum_m(\alpha_m\alpha_{n-m} - \eta\beta_m\beta_{n-m} - (n-m)c_{-m}b_{m+n}) : -2\sqrt{2}inc_{-n}\alpha_n\Big) - ac_0, \qquad (2.45)$$

where

$$\begin{aligned}
\alpha_0 &= \frac{1}{\sqrt{\pi}}p_\phi,\\
\beta_0 &= \frac{1}{\sqrt{\pi}}p_\zeta,
\end{aligned} \qquad (2.46)$$

and a is a c-number (which we call 'intercept') which accounts for the normal ordering ambiguities in Q_1. After some algebra, one finds,

$$Q_1^2 = 0 \qquad \Rightarrow \qquad a = 0 \qquad (2.47)$$

The physical state condition is given by

$$Q_1|\Phi >= 0. \tag{2.48}$$

We will show that the states in the Fock space satisfying (2.48) can be written as

$$|\Phi > = |\Phi >_{\rm phys} + Q_1|J > \tag{2.49}$$

where $|\Phi >_{\rm phys}$ contains only right-moving modes. Let us first consider the 'Minkowski case' η=1.

Following ref.[10] let us expand Q_1 in the zero modes of the ghosts,

$$Q_1 = c_0 H + Q_B + b_0 M, \tag{2.50}$$

where

$$\begin{aligned} H &= \frac{1}{2}(\alpha_0^2 - \beta_0^2) + N \\ &= \frac{1}{2}(\alpha_0^2 - \beta_0^2) + \sum_{n=1}^{\infty}(\alpha_{-m}\alpha_m - \beta_{-m}\beta_m + m(c_{-m}b_m + b_{-m}c_m)), \end{aligned} \tag{2.51}$$

$$Q_B = \frac{1}{2}\sum_{\substack{n,m\neq 0 \\ n+m\neq 0}} : c_{-n}(\alpha_m\alpha_{n-m} - \beta_m\beta_{n-m} + (n-m)c_{-m}b_{m+n}) : -\sqrt{2}i\sum nc_{-n}\alpha_n, \tag{2.52}$$

$$M = -\sum_n n : c_{-n}c_n : . \tag{2.53}$$

In (2.51) N is the number operator. From (2.45), (2.51), (2.52) and (2.53) one can verify that,

$$[Q, H] = [Q_B, H] = [Q_B, M] = [H, M] = 0, \tag{2.54}$$

and consequently, from (2.47) and (2.50):

$$Q_B^2 = -MH. \tag{2.56}$$

Let us consider states $|\Psi>$ which are built out of left-moving modes only and that satisfy the physical state condition (2.48),

$$Q_1|\Psi>=0. \tag{2.57}$$

To prove (2.49) we just have to show that, excluding the vacuum state,

$$|\Psi>=Q_1|J>, \tag{2.58}$$

i.e., that all the left-moving states satisfying the physical state condition are pure gauge or, in other words, the Fock space of left-moving states is cohomologically trivial respect to the operator Q_1. We choose our vacuum $|0>$ to be annihilated by the ghost zero mode b_0:

$$b_0|0>=0, \tag{2.59}$$

therefore the ghost zero mode entering in $|\Phi>_{phys}$ and $|\Psi>$ is c_0. Before proving (2.58) let us make the following observation. Let us assume that $|\Psi>$ is not annihilated by H, *i.e.*, $H|\Psi>\neq 0$. Then it is simple to verify that if $|\Psi>$ satisfies (2.57), (2.58) holds with

$$|J>=b_0H^{-1}|\Psi>, \tag{2.60}$$

and so we can restrict ourselves to prove (2.58) for states that besides (2.57) satisfy

$$H|\Psi>=0. \tag{2.61}$$

The observation (2.60) has been motivated by work on string field theory in the Siegel gauge[13]. Due to the choice (2.59) the states have the general form

$$|\Psi>=|\psi>+c_0|\tilde{\psi}>, \tag{2.62}$$

where $|\psi>$ and $|\tilde{\psi}>$ do not contain ghost zero modes. The physical state condition (2.57) together with (2.61) is equivalent to the following set of conditions:

$$H|\psi> = 0, \tag{2.63a}$$

$$Q_B|\psi> = 0, \tag{2.63b}$$

$$H|\tilde{\psi}> = 0, \tag{2.63c}$$

$$Q_B|\tilde{\psi}> = 0, \tag{2.63d}$$

$$M|\tilde{\psi}> = 0. \tag{2.63e}$$

We will show that the states $|\psi>$ satisfying (2.63a-b) can be written as

$$|\psi>= Q_B|j>, \tag{2.64a}$$

with

$$H|j>= 0, \tag{2.64b}$$

and that the states $|\tilde{\psi}>$ satisfying (2.63c-e) can be written in the form

$$|\tilde{\psi}>= Q_B|\tilde{j}>, \tag{2.65a}$$

with

$$H|\tilde{j}> = 0, \tag{2.65b}$$

$$M|\tilde{j}> = 0. \tag{2.65c}$$

Then the state $|J>$ in (2.58) is just

$$|J>= |j> -c_0|\tilde{j}> . \tag{2.66}$$

We will concentrate first in proving (2.64a-b). Obviously, the proof leading to (2.65a-b) is the same. However, in this case one must also show that if $|\tilde{\psi}>$ satisfies (2.63e) then (2.65c) holds. We start defining:

$$p_0^{\pm} = \frac{1}{\sqrt{2}}(\alpha_0 \pm \beta_0), \tag{2.67a}$$
$$p_n^{\pm} = \frac{1}{\sqrt{2}}(\alpha_n \pm \beta_n). \tag{2.67b}$$

From (2.44) and (2.46) one has

$$[p_n^+, p_m^-] = n\delta_{n+m}, \tag{2.68a}$$

$$[p_n^+, p_m^+] = [p_n^-, p_m^-] = 0. \tag{2.68b}$$

Following ref.[10] we rescale the oscillator operators by an appropriate power of a real parameter κ,

$$\begin{aligned}
p_0^+ &\longrightarrow \frac{1}{\kappa}p_0^+,\\
p_0^- &\longrightarrow \kappa p_0^-,\\
b_n &\longrightarrow \frac{1}{\kappa}b_n, \quad n \neq 0,\\
c_n &\longrightarrow \kappa c_n, \quad n \neq 0.
\end{aligned} \tag{2.69}$$

This rescaling is such that the commutation relations in (2.68) and the ghost commutation relations in (2.44) remain unchanged. The operator H remains also unchanged. However, the operator Q_B gets the following expansion:

$$Q_B(\kappa) = A + \kappa B + \kappa^2 C, \tag{2.70}$$

where

$$A = \sum_{n\neq 0} c_{-n} p_0^+ p_n^-, \tag{2.71a}$$

$$B = \sum_{\substack{n\neq 0\\ n-m\neq 0}} :c_{-n}(p^+_m p^-_{n-m} + p^-_m p^+_{n-m}):$$

$$-\frac{1}{2}\sum_{\substack{n,m\neq 0\\ n+m\neq 0}} (n-m):c_{-n}c_{-m}b_{n+m}: -i\sum nc_{-n}(p^+_n + p^-_n), \tag{2.71b}$$

$$C = \sum_{n\neq 0} c_{-n}p^-_0 p^+_n. \tag{2.71c}$$

Notice that the term in Q_B originated from the last term in the modified Siegel lagrangian (2.34) enters in the operator B in the expansion (2.70). Since it does not enter into the operators A or C, the proof of the no-ghost theorem follows the lines of ref.[10].

From (2.56), (2.70) and the fact that under the rescaling (2.69) the operator H remains unchanged but the operator M gets a factor κ^2 one obtains the following useful identities:

$$A^2 = 0, \tag{2.72a}$$

$$\{A, B\} = 0, \tag{2.72b}$$

$$B^2 + \{A, C\} = -MH, \tag{2.72c}$$

$$\{B, C\} = 0, \tag{2.72d}$$

$$C^2 = 0. \tag{2.72e}$$

The physical state conditions on the left-moving sector (2.63a-b) become now

$$H|\psi(\kappa)>=0, \tag{2.73a}$$

$$Q_B(\kappa)|\psi(\kappa)>=0, \tag{2.73b}$$

where $Q_B(\kappa)$ is given in (2.70) and $|\psi(\kappa)>$ has the power expansion

$$|\psi(\kappa)>=\sum_{n=0}^{\infty}\kappa^n|\psi_n>, \tag{2.74}$$

i.e. for each term of the power expansion of (2.74) one has the equations

$$H|\psi_n>=0, \tag{2.75a}$$

$$A|\psi_n>+B|\psi_{n-1}>+C|\psi_{n-2}>=0. \tag{2.75b}$$

Equations (2.64a-b) now become

$$|\psi(\kappa)>=Q_B(\kappa)|j(\kappa)>, \tag{2.76a}$$

$$H|\psi(\kappa)>=0, \tag{2.76b}$$

or

$$|\psi_n>=A|j_n>+B|j_{n-1}>+C|j_{n-2}>, \tag{2.77a}$$

$$H|\psi_n>=0, \tag{2.77b}$$

where we have used the fact that $|j>$ has the expansion

$$|j>=\sum_{n=0}^{\infty}\kappa^n|j_n>. \tag{2.78}$$

In equations (2.75) and (2.77) it is understood that $|\psi_n>=|j_n>=0$ for $n<0$.

Our goal is to show that for arbitrary κ the states satisfying (2.73) (or (2.75)) must be of the form (2.76) (or (2.77)). Then, taking $\kappa=1$ we will have proved that all the left-moving states satisfying (2.63a-b) are of the form (2.64a-b). We perform the proof by iteration in the power of κ in the expansion (2.74).

We consider first the case $n=0$. At this order we have to show, according to (2.75) and (2.77), that any state $|\chi>$ satisfying

$$H|\chi>=0, \tag{2.79a}$$

$$A|\chi>=0, \tag{2.79b}$$

is of the form

$$|\chi> = A|k>, \tag{2.80a}$$

$$H|k> = 0, \tag{2.80b}$$

i.e., we have to show that the Fock space generated by the left-moving modes is cohomologically trivial with respect to the operator A.

Using (2.68) and (2.44) it is simple to verify that A satisfies the (anti)commutation relations

$$\begin{aligned} &[A, p_n^+] = -np_0^+ c_n, \qquad &[A, p_n^-] = 0, \\ &\{A, b_n\} = p_0^+ p_n^-, \qquad &\{A, c_n\} = 0. \end{aligned} \tag{2.81}$$

Let us consider states for which $p_0^+ \neq 0$ and let us introduce the operator

$$R_1 = \frac{1}{p_0^+} \sum_{n>0} \frac{1}{n} (b_{-n} P_0 p_n^+ + p_{-n}^+ P_0 b_n), \tag{2.82}$$

where

$$P_0 = |0><0|, \tag{2.83}$$

and $|0>$ is the vacuum state. We have

$$\begin{aligned} P_1 &= \{R_1, A\} \\ &= \sum_{n>0} (\frac{1}{n} p_{-n}^+ P_0 p_n^+ + \frac{1}{n} p_{-n}^- P_0 p_n^+ + b_{-n} P_0 c_n + c_{-n} P_0 b_n). \end{aligned} \tag{2.84}$$

The operator P_1 can be identified with the projector on the one-particle unphysical states $p_{-n}^{\pm}|0>$, $b_{-n}|0>$ and $c_{-n}|0>$. Any combination of these one-particle states $|\chi_1>$ can be written as

$$|\chi_1 >= P_1|\chi_1 >= \{R_1, A\}|\chi_1 > . \tag{2.85}$$

Similarly, one introduces the operator

$$R_N = \frac{1}{Np_0^+} \sum_{n>0} \frac{1}{n}(b_{-n} P_{N-1} p_n^+ + p_{-n}^+ P_{N-1} b_n), \tag{2.86}$$

leading to the N-particle state projector

$$\begin{aligned} P_N &= \{R_N, A\} \\ &= \frac{1}{N} \sum_{n>0} (\frac{1}{n} p_{-n}^+ P_{N-1} p_n^+ + \frac{1}{n} p_{-n}^- P_{N-1} p_n^+ + b_{-n} P_{N-1} c_n + c_{-n} P_{N-1} b_n). \end{aligned} \tag{2.87}$$

The projector on the Fock space generated by $p_{-n}^{\pm}$, b_{-n} and c_{-n} is therefore

$$P = P_0 + \sum_{N=1}^{\infty} P_N = P_0 + \{A, R\}, \tag{2.88}$$

where

$$R = \sum_{N=1}^{\infty} R_N, \tag{2.89}$$

and so, any state $|\chi>$ can be written as

$$|\chi >= P_0 |\chi > + \{A, R\} |\chi > . \tag{2.90}$$

This formula implies that this Fock space is cohomologically trivial respect to the operator A:

$$A|\chi >= 0 \quad \Rightarrow \quad |\chi >= P_0 |\chi > + A(R|\chi >) \tag{2.91}$$

Finally, to complete the proof of our statement we have to show (2.80b) *i.e.* that $H|k >= 0$. This is trivial from (2.90) and the fact that

$$[H, P] = 0. \tag{2.92}$$

The result (2.91) has been obtained for states on which $p_0^+ \neq 0$. For states on which $p_0^+ = 0$ but $p_0^- \neq 0$ one can follow the same steps after interchanging $p_0^- \leftrightarrow p_0^+$ in the rescaling

(2.69). In this way one obtains $A \leftrightarrow C$ in (2.71) and the corresponding R_N operators in (2.86) contain $1/p_0^-$ instead of $1/p_0^+$. To account for all the cases just notice that the only state on which both p_0^+ and p_0^- vanish and (2.63a) holds is the vacuum state as can be concluded from (2.51) and the fact that the operator N is positive definite.

Let us remark that it is essential in the analysis of the cohomology of A that there is a KO quartet α_n, β_n, c_n and b_n. Suppose, for example, that β_n is absent, then equation (2.84) does not hold. One has instead

$$\{R_1, A\} = P_1^{(\alpha)} + \frac{1}{2}P_1^{ghost}. \tag{2.93}$$

where $P_1^{(\alpha)}$ is the α-dependent part of P_1 in (2.84) and P_1^{ghost} the ghost-dependent one. The one-particle states are cohomologically trivial. However, as a consequence of (2.93) the two-particle state (for example),

$$|\chi_2 >= (\alpha_{-1}\alpha_{-2} - b_{-1}c_{-2} - b_{-2}c_{-1})|0 >, \tag{2.94}$$

which satisfies $A|\chi_2 >= 0$, is not of the form $|\chi_2 >= A|k >$, and it has negative norm.

We have shown that for $n = 0$ the solution to (2.75) is of the form predicted in (2.77). Let us assume that this is true for n, $n < N$. We will show now that then the solution to (2.75) is also of the form (2.77) for $n = N$. The equation that we must solve is (from (2.75)):

$$A|\psi_N > +B|\psi_{N-1} > +C|\psi_{N-2} >= 0. \tag{2.95}$$

Using the induction hypothesis this equation becomes

$$\begin{aligned} A|\psi_N > &+ B(A|j_{N-1} > +B|j_{N-2} > +C|j_{N-2} >) \\ &+ C(A|j_{N-2} > +B|j_{N-3} > +C|j_{N-4} >) = 0, \end{aligned} \tag{2.96}$$

which, after making use of (2.72) becomes,

$$A(|\psi_N > -B|j_{N-1} > -C|j_{N-2} >) - MH|j_{N-2} >= 0. \tag{2.97}$$

The last term of this equation vanishes after using the induction hypothesis in (2.77b). The rest is just the operator A acting on a state. Using (2.91) one finds

$$|\psi_N> -B|j_{N-1}> -C|j_{N-2}>= A|j_N>, \tag{2.98}$$

and therefore the fact that (2.77) is the solution to (2.75) is proven. To complete the proof we must show that $H|j_N>= 0$. This follows from the fact that, as one concludes from (2.54) and (2.70),

$$[H, A] = [H, B] = [H, C] = 0. \tag{2.99}$$

We have shown that the states $|\psi>$ satisfying (2.63a-b) are of the form (2.64a-b). The proof concerning the states $|\tilde{\psi}>$ satisfying (2.63c-d) leading to (2.65a-b) follows the same steps. To prove (2.65c) one must use the fact that

$$[M, A] = [M, B] = [M, C] = [M, P] = 0. \tag{2.100}$$

This concludes our analysis of the left-moving sector. We have shown that (2.57) implies (2.58). Going back to (2.48) *i.e.*, to states which involve both sectors, we observe that (2.58) implies (2.49). The states $|\Phi>_{\rm phys}$ contain only right-moving modes and, according to our choice of vacuum, (2.59), the ghost zero mode c_0. The general form of $|\Phi>_{\rm phys}$ is

$$|\Phi>_{\rm phys}= |\vartheta>_{\rm phys} + c_0|\tilde{\vartheta}>, \tag{2.101}$$

where $|\vartheta>_{\rm phys}$ and $|\tilde{\vartheta}>$ contain only right-moving modes. There are several possibilities to decouple the states in $|\tilde{\vartheta}>$ from the physical sector[14]. The standard choice[10] is to define the inner product among states $|\Phi>_{\rm phys}$ with a c_0 insertion,

$$_{\rm phys}< \Phi|c_0|\Phi>_{\rm phys}=_{\rm phys}< \vartheta|\vartheta>_{\rm phys}\ . \tag{2.102}$$

Another possibility is to restrict the states of the theory to have the same ghost number as the vacuum state.

We have shown that the Q-physical states of this theory are formed by the right-moving parts of ϕ and ζ, and by the null left-moving momenta, as it follows from (2.63). To describe only right-moving bosons in the 'Minkowski case' this null left-moving momenta must be constrained to be zero. Let us now consider the 'Euclidean case', $\eta = -1$, *i.e.* the case in which the two fields ϕ and ζ have the same sign in their kinetic terms. In this case H in (2.51) is positive definite and equations (2.63) are satisfied only by the ground state and by the right-moving states. All other states not satisfying (2.63a) are cohomologically trivial, according to (2.60). We argue that the fact that in both the 'Minkowski case' and the 'Euclidean case' the no-ghost theorem holds is related to the possibility of 'Wick rotating' the momentum p_ζ of ζ.

Let us show at this point that for $n \neq 2$ the BRST quantized Siegel action does not describe chiral bosons. For $n = 1$ the equation (2.63a) is never satisfied, because the 'intercept' is negative and therefore H has no zero eigenvalues. For $n > 2$ the 'intercept' is positive, and (2.63a) has solutions only when the left-moving momenta are different from zero and therefore the theory does not describe right-moving scalars. Another argument for neglecting the case $n > 2$ is that in the 'Minkowski case' the physical states will contain $n-2$ left-moving modes since only two modes will form a KO quartet with the ghosts b_{++} and c^+. Therefore this case does not describe chiral bosons. An analogous argument involving the determinants in the partition function leads to the same conclusions also in the 'Euclidean case'. To describe $2n$ chiral bosons one can introduce n gauge fields.

Finally, let us make some comments on the coupling of the modified Siegel lagrangian to gravity. One expects that the Virasoro generators corresponding to the gravity field would be

$$
\begin{aligned}
T_{++} &= L_{++} + iB_{++}\partial_+ C^+ \\
&= (\partial_+\phi)^2 - \eta(\partial_+\zeta)^2 - 2ib_{++}\partial_+ c^+ + \frac{2}{\sqrt{2\pi}}\partial_+\partial_+\phi + iB_{++}\partial_+ C^+ \qquad (2.106)
\end{aligned}
$$

$$
T_{--} = L_{--} + iB_{--}\partial_- C^- = (\partial_-\phi)^2 - \eta(\partial_-\zeta)^2 + iB_{--}\partial_- C^- \qquad (2.107)
$$

where $B_{\pm\pm}$ and $C^{\pm}$ are the ghost fields corresponding to the gravity field. The Virasoro algebra for T_{++} has a central charge -26, coming from B_{++} and C^+, since L_{++} has central charge zero. The Virasoro algebra for T_{--} has a central charge -24. This is the expected contribution to the central charge due to two chiral bosons. Notice the presence of the last but one term of T_{++} in (2.106). This term is necessary to have a closed algebra. If this term were absent the Siegel lagrangian coupled to gravity would be of the form (2.20). We do not know how to obtain this algebra from a lagrangian.

3. Quadratic Lagrangian for chiral bosons

In this section we present another formulation of two right-moving scalars which has characteristics analogous to the quantized Siegel lagrangian in the 'Minkowski case' but with a quadratic lagrangian. Consider the scalar fields $\phi(\tau,\sigma)$ and $\zeta(\tau,\sigma)$, governed by the lagrangian,

$$
\mathcal{L}_2 = -\partial_+\phi\partial_-\phi + \partial_+\zeta\partial_-\zeta + \lambda^+\partial_+(\phi-\zeta), \qquad (3.1)
$$

where $\lambda^+(\tau,\sigma)$ is a gauge field Lagrange multiplier. This lagrangian is invariant under the transformations,

$$\begin{aligned}\delta\phi &= \epsilon,\\ \delta\zeta &= \epsilon,\\ \delta\lambda^{+} &= 2\partial_{-}\epsilon.\end{aligned} \tag{3.2}$$

The generators of the symmetry transformations on ϕ and ζ are given by the first-class constraints

$$\partial_{+}(\phi-\zeta)\approx 0. \tag{3.3}$$

The lagrangian (3.1) and the symmetry (3.2) are analogous to a gauged Wess-Zumino model and an abelian Kac-Moody symmetry respectively.

After BRST-quantizing (3.1) one obtains

$$\mathcal{L}_{2}^{q} = -\partial_{+}\phi\partial_{-}\phi + \partial_{+}\zeta\partial_{-}\zeta + ib_{+}\partial_{-}c, \tag{3.4}$$

which is the same quantum lagrangian as in (2.39). However, the BRST transformations are different:

$$\begin{aligned}\delta\phi &= \Lambda c,\\ \delta\zeta &= \Lambda c,\\ \delta c &= 0,\\ \delta b_{+} &= -2i\Lambda\partial_{+}(\phi-\zeta).\end{aligned} \tag{3.5}$$

As we will show below, the left-moving part of ϕ and ζ form together with the ghosts b_{+} and c a KO quartet (ref.[9]). The physical state condition is

$$Q_{2}|\Phi>=0. \tag{3.6}$$

where Q_2 is the nilpotent BRST operator

$$Q_{2} = \frac{1}{\sqrt{2}}\int d\sigma c(\sigma)\partial_{+}(\phi(\sigma)-\zeta(\sigma)), \tag{3.7}$$

The Fock space is made out of the creation operators associated to ϕ, ζ, b_+ and c. The Fourier expansions of our fields are just the ones in (2.43) and the oscillator operators satisfy the (anti)commutation relations (2.44). Using the expansions (2.67), (2.46), (2.43) and (3.7) one finds that the BRST operator can be written in the form

$$Q_2 = c_0 p_0^- + \sum_{n\neq 0} c_{-n} p_n^- . \tag{3.8}$$

The operator Q_2 has a much simpler form in this formulation as compared to the one in the previous section. It has the form of the quadratic nilpotent operator A in (2.71a). Notice also that there are not normal-order ambiguities in Q_2 and so there is no need for an 'intercept'-like term in (3.8).

As before, we choose our vacuum $|0>$ to be annihilated by b_0,

$$b_0|0>= 0. \tag{3.9}$$

The physical state condition,

$$Q_2|\Phi>= 0, \tag{3.10}$$

is equivalent to the conditions

$$(p_\phi - p_\zeta)|\varphi> = 0, \tag{3.11a}$$

$$A_2|\varphi> = 0, \tag{3.11b}$$

$$A_2|\tilde{\varphi}> = 0, \tag{3.11c}$$

where

$$|\Phi>= |\varphi> + c_0|\tilde{\varphi}>, \tag{3.12}$$

and

$$A_2 = \sum_{n \neq 0} c_{-n} p_n^- . \tag{3.13}$$

Condition (3.11a) says that the left-moving momenta of ϕ and ζ are equal ($p_\phi = p_\zeta$). On the other hand, from the fact that

$$\begin{aligned} [A_2, p_n^+] &= -nc_n, & \{A_2, c_n\} &= 0, \\ \{A_2, p_n^-\} &= 0, & [A_2, b_n] &= p_n^-, \end{aligned} \tag{3.14}$$

the operators α_n, β_n, b_n and c_n form a KO quartet. Using the same arguments as in the previous section led to (2.94) one obtains,

$$|\Phi >= P_{(0)}|\Phi > +Q_2|J >, \tag{3.15}$$

where $P_{(0)}$ is the projector onto states made out of oscillators in the right-moving sector. Notice that due to the simplicity of Q_2 the proof of the absence of ghost in the left-moving sector, *i.e.*, of (3.15), is much simpler for this formulation than for the one in the previous section. As in that case, (3.1) describes two chiral bosons once one puts to zero the momentum p_ζ.

Contrary to the case of the modified Siegel lagrangian, it is straightforward to couple the lagrangian (3.1) to gravity:

$$\mathcal{L}_2^h = -\sqrt{h} h^{\alpha\beta} \Big(\partial_\alpha \phi \partial_\beta \phi - \partial_\alpha \zeta \partial_\beta \zeta \Big) + \sqrt{h} e_+^\alpha \lambda^+ \partial_\alpha (\phi - \zeta). \tag{3.16}$$

It is simple to observe that this lagrangian gives the correct contribution of two chiral bosons to the conformal anomaly. In the left sector one has a contribution $1 + 1$ from the fields ϕ and ζ and -2 from the $U(1)$ field λ^+. In the right sector one has a contribution $1 + 1$ from the scalar fields. Of course, in addition to this, there is a contribution -26 from the gravity field $h^{\alpha\beta}$ in each sector.

The lagrangian (3.1) can be obtained from a general procedure[7] to transform second-class constraints (in this case the self-duality constraint $\partial_+\phi \approx 0$) into first-class constraints (in this case constraint (3.3)). Let us briefly review the formalism of ref.[7].

Batalin and Fradkin have shown that given a set of canonically conjugate variables p_i, q^i,

$$[q^i, p_j] = i\delta^i_j, \tag{3.17}$$

subjected to second-class constraints,

$$T_\alpha(p_i, q^j) \approx 0, \tag{3.18}$$

$$[T_\alpha, T_\beta]_\pm = iD_{\alpha\beta}, \tag{3.19}$$

where $D_{\alpha\beta}$ is an operator that may depend on p_i and q^i, it can be transformed into a system with first-class constraints in an enlarged phase-space, containing auxiliary canonical variables ξ^α and η_α, in correspondence with the original second-class constraints

$$[\xi^\alpha, \eta_\beta]_\pm = -i\delta^\alpha_\beta. \tag{3.20}$$

These first-class constraints are

$$\hat{T}_\alpha = T_\alpha + V^\beta_\alpha(\eta_\beta + \frac{1}{2}\omega_{\beta\gamma}\xi^\gamma), \tag{3.21}$$

where $\omega_{\alpha\beta}$ is a c-number and V^β_α are operators satisfying

$$D_{\alpha\beta} = V^\gamma_\alpha \omega_{\gamma\delta} V^\delta_\beta. \tag{3.22}$$

The operators $D_{\alpha\beta}$ can be interpreted as a symplectic metric on the phase-space of p_i and q^i; $\omega_{\alpha\beta}$ is a flat symplectic metric on the enlarged phase-space; V^β_α is the vielbein which transform the symplectic metric $D_{\alpha\beta}$ into the flat symplectic metric $\omega_{\alpha\beta}$. The presence of the first-class constraints (3.21) implies that one must choose a gauge fixing. One possible gauge fixing consists of the choice

$$\Phi = \eta_\beta + \frac{1}{2}\omega_{\beta\gamma}\xi^\gamma = 0, \tag{3.23}$$

and then one reduces the situation to the original phase-space with (3.21) becoming the second-class constraints (3.18).

In the enlarged phase-space (q^i, ξ^α, p_i and η_α) one imposes the canonical second-class constraints

$$\tilde{T}_\alpha = \eta_\alpha - \frac{1}{2}\omega_{\alpha\beta}\xi^\beta \approx 0. \tag{3.24}$$

These constraints are canonical in the sense that they involve a linear combination of the canonically conjugate variables ξ^α and η_β with c-number coefficients. The advantage of the formulation dealing with second-class constraints by Batalin and Fradkin[7] is that while the second-class constraint (3.18) can be an involved function of p_i and q^i, the second-class constraint (3.24) is in canonical form.

For the case of a single chiral boson in $D = 2$ the self-duality second-class constraint (2.2) is already in canonical form. However, let us follow the procedure described in ref.[7] to analyze the system in the enlarged phase-space. The new first and second-class constraints are, respectively,

$$\hat{T}(\sigma) = \pi(\sigma) + \phi'(\sigma) + \eta(\sigma) + \xi'(\sigma), \tag{3.25}$$

$$\tilde{T}(\sigma) = \eta(\sigma) - \xi'(\sigma), \tag{3.26}$$

where $\eta(\tau,\sigma)$ and $\xi(\tau,\sigma)$ are the auxiliary canonically conjugate fields (3.20),

$$[\xi(\sigma), \eta(\sigma')] = -i\delta(\sigma - \sigma') \tag{3.27}$$

Following ref.[7] we introduce the canonically conjugate ghost fields $b(\tau,\sigma)$ and $c(\tau,\sigma)$,

$$\{b(\sigma), c(\sigma')\} = \delta(\sigma - \sigma'), \tag{3.28}$$

and we find the BRST operator

$$Q = \int d\sigma c(\sigma)\hat{T}(\sigma), \tag{3.29}$$

and the unitarizing hamiltonian

$$H^\Psi = \int d\sigma\Big(\frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2 - \frac{1}{2}\eta^2 - \frac{1}{2}\xi'^2 + \tilde{T}(\eta + \xi') - \frac{1}{2}(\eta + \xi')^2\Big) + \{\Psi, Q\}, \tag{3.30}$$

where Ψ is a gauge-fixing fermionic function. Using (2.3) and (3.25-28) one finds,

$$[\hat{T}(\sigma), \hat{T}(\sigma')] = 0, \tag{3.31}$$

$$Q^2 = 0, \tag{3.32}$$

$$[H^\Psi, Q] = 0. \tag{3.33}$$

Comparing this hamiltonian with the hamiltonian H_2 that one obtains from the quadratic lagrangian (3.1) of sect. 3, we find

$$H^\Psi - H_2^{\Psi_2} = \frac{1}{4}\int d\sigma(\eta - \xi')^2, \tag{3.34}$$

where,

$$\Psi - \Psi_2 = -\frac{i}{2}\int d\sigma b(\eta + \xi')^2. \tag{3.35}$$

Notice that the BRST operator is the same for both theories. Therefore the two formulations are equivalent if one imposes the constraint (3.26).

Notice that it is not possible to write down a relativistic covariant quantum lagrangian describing a single self-dual scalar; any attempt to remove a second-class self-duality constraint on ϕ by squaring it to get a first-class constraint, as in sect. 2, or adding an extra field to transform the second-class constraint (2.4) into the first-class constraint (3.3), as in this section, leads to a second-class constraint imposed on the extra field.

4. Conclusion

We have considered two approaches to the quantization of self-dual fields in two dimensions. Siegel's cubic lagrangian for a single chiral boson has an anomaly which can be eliminated introducing a 'Liouville term'. To BRST quantize Siegel's lagrangian it is necessary to introduce an extra scalar. We prove the no-ghost theorem, which says that the theory describes two chiral bosons, both when their kinetic terms have the same and the opposite sign; in the latter case, one must add the constraint that the left-moving momentum p_ζ of ζ be zero. A similar result can be obtained from the application of the generalized canonical formalism for second-class constraints of Batalin and Fradkin, leading to a quadratic lagrangian for two chiral bosons with the opposite sign in their kinetic terms. The fact that it is not possible to transform a single self-duality constraint into a first-class constraint can be understood in terms of the path integral[15]. For the case of the chiral boson the partition function takes the form

$$Z = \int [d\phi][d\pi]\big(\det{}^{\frac{1}{2}}[T(\sigma), T(\sigma')]\big)\delta(\pi + \phi')\mathrm{exp}\Big(i\int d\tau d\sigma\big(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}\phi'^2\big)\Big).$$

where $T(\sigma)$ are the second class constraints (2.4). The square root of the determinant can not be substituted by a Faddeev-Popov determinant; only pairing two square roots of determinants one can obtain a Faddeev-Popov determinant, and therefore a first-class constraint.

The work presented in this paper leads to a variety of investigations. The coupling of the modified Siegel's action to auxiliary gravity could be studied and its role in the analysis of the quantum behavior of chiral bosons on higher genus Riemann surfaces could be investigated. The quadratic action presented in this work could be used in covariant computations related to the heterotic string which do not depend on the signature of the kinetic terms of the chiral bosons. Finally, one could investigate the Batalin-Fradkin hamiltonian formulation for two self-dual fields in $D = 6$ and 10 in the 'Minkowski case'. Analogously to the analysis of sect. 3, one could then find the gravitational anomaly for two self-dual fields.

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While we were completing this work we received the preprint WIS-87/18/Mar-PH by C. Imbimbo and A. Schwimmer titled 'The lagrangian formulation of chiral bosons', in which it is also shown how to cancel the gauge anomaly; however, we disagree with their results concerning the analysis of the physical states of the theory.

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